

NOTES ON LEAST SQUARES

1. INTRODUCTION

The theory of least squares and its application to adjustment of survey measurements is well known to every geodesist. The invention of the method is generally attributed to Karl Freidrich Gauss (1777-1855) but could equally be credited to Adrien-Marie Legendre (1752-1833).

Gauss used the method of least squares to compute the elements of the orbit of the minor planet *Ceres* and predicted its position in October 1801 from a few observations made in the previous year. He published the technique in 1809 in *Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem Ambientium* (Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections), mentioning that he had used it since 1795, and also developed what we now know as the normal law of error, concluding that: "... *the most probable system of values of the quantities ... will be that in which the sum of the squares of the differences between the actually observed and computed values multiplied by numbers that measure the degree of precision, is a minimum.*" (Gauss 1809).

Legendre published an independent development of the technique in *Nouvelles méthodes pour la détermination des orbites des comètes* (New methods for the determination of the orbits of comets), Paris, 1806 and also as the "Méthod des moindres carriés" (Method of Least Squares), published in the *Mémoires de l'Institut national des sciences at arts*, vol. 7, pt. 2, Paris, 1810.

After these initial works, the topic was subjected to rigid analysis and by the beginning of the 20th century was the universal method for the treatment of observations. Merriman (1905) compiled a list of 408 titles, including 72 books, written on the topic prior to 1877 and publication has continued unabated since then. Leahy (1974) has an excellent summary of the

development of least squares and clearly identifies the historical connection with mathematical statistics, which it pre-dates.

The current literature is extensive; the books *Observations and Least Squares* (Mikhail 1976) and *Analysis and Adjustment of Survey Measurements* (Mikhail and Gracie 1981), and lecture notes by Cross (1992), Krakiwsky (1975) and Wells and Krakiwsky (1971) stand out as the simplest modern treatments of the topic.

Following Wells and Krakiwsky (1971, pp.8-9), it is interesting to analyse the following quotation from Gauss' *Theoria Motus* (Gauss, 1809, p.249).

"If the astronomical observations and other quantities, on which the computation of orbits is based, were absolutely correct, the elements also, whether deduced from three or four observations, would be strictly accurate (so far indeed as the motion is supposed to take place exactly according to the laws of KEPLER), and, therefore, if other observations were used, they might be confirmed, but not corrected. But since all our measurements and observations are nothing more than approximations to the truth, the same must be true of all calculations resting upon them, and the highest aim of all computations made concerning concrete phenomena must be to approximate, as nearly as practicable, to the truth. But this can be accomplished in no other way than by a suitable combination of more observations than the number absolutely requisite for the determination of the unknown quantities. This problem can only be properly undertaken when an approximate knowledge of the orbit has been already attained, which is afterwards to be corrected so as to satisfy all the observations in the most accurate manner possible."

This single paragraph, written almost 200 years ago, embodies the following concepts, which are as relevant today as they were then.

- (i) Mathematical models may be incomplete,
- (ii) Physical measurements are inconsistent,
- (iii) All that can be expected from computations based on inconsistent measurements are estimates of the "truth",
- (iv) Redundant measurements will reduce the effect of measurement inconsistencies,

- (v) Initial approximations to the final estimates should be used, and finally,
- (vi) Initial approximations should be corrected in such a way as to minimise the inconsistencies between measurements (by which Gauss meant his method of least squares).

These notes contain a development of Least Squares processes applicable to surveying and geodesy. Examples and exercises of least squares processes are given using MATLAB, an interactive, matrix-based system for scientific and engineering computation and visualization. The name MATLAB is derived from MATrix LABoratory and is licensed by The MathWorks, Inc.